

Sharp endpoint estimates for some operators associated with the Laplacian with drift in Euclidean space

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Abstract. Let $v \neq 0$ be a vector in \mathbb{R}^n . Consider the Laplacian on \mathbb{R}^n with drift $\Delta_v = \Delta + 2v \cdot \nabla$ and the measure $d\mu(x) = e^{2\langle v, x \rangle} dx$, with respect to which Δ_v is self-adjoint. This measure has exponential growth with respect to the Euclidean distance. We study weak type $(1, 1)$ and other sharp endpoint estimates for the Riesz transforms of any order, and also for the vertical and horizontal Littlewood-Paley-Stein functions associated with the heat and the Poisson semigroups.

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1 Introduction

Consider the weighted manifold $\mathbb{R}^{(n,v)}$, defined as \mathbb{R}^n with the Euclidean distance and the measure $d\mu(x) = e^{2\langle v, x \rangle} dx$. Here $v = (v_1, \dots, v_n) \in \mathbb{R}^n \setminus \{0\}$ is fixed and $\langle \cdot, \cdot \rangle$ denotes the inner product on \mathbb{R}^n . Notice that large balls in this space have exponential volume growth. With $\mathbb{R}^{(n,v)}$, we associate the Laplacian with drift

$$\Delta_v = \sum_{i=1}^n \left(\frac{\partial^2}{\partial x_i^2} + 2v_i \frac{\partial}{\partial x_i} \right).$$

We denote by ∇ the usual gradient operator in \mathbb{R}^n . Notice that the Green formulas holds with respect to the measure μ , that is,

$$\int_{\mathbb{R}^n} f \Delta_v w \, d\mu = - \int_{\mathbb{R}^n} \langle \nabla f, \nabla w \rangle \, d\mu = \int_{\mathbb{R}^n} w \Delta_v f \, d\mu,$$

provided f and w are smooth and f or w has compact support. Thus Δ_v has a selfadjoint extension in $L^2(\mathbb{R}^n, d\mu)$.

We shall consider Riesz transforms and Littlewood-Paley-Stein functions of any order in $\mathbb{R}^{(n,v)}$. These operators are defined and studied in many general settings, such as Lie groups, symmetric spaces and other Riemannian manifolds. Their L^p boundedness properties for $1 < p < \infty$ have been well studied and are known in several cases. We refer the reader to [1], [5]-[12], [18], [21]-[25] and references therein. In particular, the classical results are not always valid on manifolds. Interesting counterexamples can be found in [18], [9], [7] and [8] for Riesz transforms of order one, and in [14], [1] and [12] for those of order two. The setting of the Ornstein-Uhlenbeck semigroup in Euclidean space is considered in [13] and [26]; see also [2]. There the measure is Gaussian, thus finite but not doubling.

The weak type $(1,1)$ property of these operators is more difficult and less known in spaces of exponential volume growth. The main reason is that the existing methods to treat singular integrals are insufficient. Theorem 1.2 of [9] is a weaker estimate. But some results have been established; see [14] and [27] for the affine group, which is a typical case without spectral gap. Some other groups and spaces are treated in [3], [4] and [17]. Further, [20] and [19] deal with the Laplacian with drift.

Returning to our setting $\mathbb{R}^{(n,v)}$, we mention that Lohoué and Mustapha [25] proved that the Riesz transforms $\nabla^k (-\Delta_v)^{-k/2}$ of any order k are bounded on L^p , $1 < p < \infty$. Their setting and results are actually more general. In [20], the authors and Y.-R. Wu showed that the first-order Riesz transform $\nabla (-\Delta_v)^{-1/2}$ is of weak type $(1,1)$ in $\mathbb{R}^{(n,v)}$. Here and in the sequel, L^p and weak L^p estimates in $\mathbb{R}^{(n,v)}$ always refer to the measure μ .

For $\mathbb{R}^{(n,v)}$ we observe that $\partial/\partial x_i$ commutes with Δ_v and thus with any negative power of $-\Delta_v$, so that the factors ∇^k and $(-\Delta_v)^{-k/2}$ can be written in any order. We will study the weak type $(1,1)$ property of $\nabla^k (-\Delta_v)^{-k/2}$. But instead of ∇^k , we will use a general homogeneous differential operator of order $k \geq 1$,

$$D = \sum_{|\alpha|=k} a_\alpha \partial^\alpha \tag{1.1}$$

with constant coefficients, not all 0. Our Riesz operator will thus be

$$R_D = D(-\Delta_v)^{-\frac{k}{2}}.$$

Letting ∂_v denote differentiation along the vector v , we can write D as a sum

$$D = \sum_{i=0}^k \partial_v^i D'_{k-i},$$

where D'_{k-i} is a constant coefficient operator of order $k-i$ involving only differentiation in directions orthogonal to v . The maximal order of differentiation along v is then

$$q = \max \{i : D'_{k-i} \neq 0\} \in \{0, \dots, k\},$$

and this quantity turns out to be significant.

Our result about R_D is the following.

Theorem 1 *With D and q as just described, the Riesz transform $R_D = D(-\Delta_v)^{-k/2}$ is of weak type $(1, 1)$ if and only if $q \leq 2$. When $q \geq 3$, there exists a constant $C = C(v, D)$ such that for all $f \in L(1 + \ln^+ L)^{\frac{q}{2}-1}(\mu)$ and all $\lambda > 0$, we have*

$$\mu \{x; |R_D f(x)| > \lambda\} \leq C \int \frac{|f|}{\lambda} \left(1 + \ln^+ \frac{|f|}{\lambda}\right)^{\frac{q}{2}-1} d\mu. \quad (1.2)$$

This inequality is sharp in the sense that q cannot be replaced by any smaller number.

Let $(e^{t\Delta_v})_{t>0}$ denote the heat semigroup on $\mathbb{R}^{(n,v)}$, which is a symmetric diffusion semigroup in the sense of [28] (the conservation property can be justified by Theorem 11.8 in [16], and the other properties are obvious). Further, let $(e^{-t\sqrt{-\Delta_v}})_{t>0}$ denote the Poisson semigroup. For $f \in C_0^\infty$, we define the vertical Littlewood-Paley-Stein functions associated with the operator D from (1.1) as

$$\mathcal{H}_D(f)(x) = \left(\int_0^{+\infty} \left| t^{\frac{k}{2}} D e^{t\Delta_v} f(x) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

and

$$\mathcal{G}_D(f)(x) = \left(\int_0^{+\infty} \left| t^k D e^{-t\sqrt{-\Delta_v}} f(x) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

The L^p ($1 < p < +\infty$) boundedness of \mathcal{G}_D and \mathcal{H}_D is easy to verify.

Theorem 2 (a) *The operator \mathcal{H}_D is of weak type $(1, 1)$ if and only if $q \leq 1$. When $q > 1$, there exists a constant $C = C(v, D)$ such that for all $f \in L(1 + \ln^+ L)^{\frac{q}{2}-\frac{3}{4}}(\mu)$ and all $\lambda > 0$*

$$\mu \{x; \mathcal{H}_D f(x) > \lambda\} \leq C \int \frac{|f|}{\lambda} \left(1 + \ln^+ \frac{|f|}{\lambda}\right)^{\frac{q}{2}-\frac{3}{4}} d\mu. \quad (1.3)$$

(b) The operator \mathcal{G}_D is of weak type $(1, 1)$ if and only if $q \leq 2$. When $q > 2$, there exists a constant $C = C(v, D)$ such that for all $f \in L(1 + \ln^+ L)^{\frac{q}{2}-1}(\mu)$ and all $\lambda > 0$

$$\mu \{x; \mathcal{G}_D f(x) > \lambda\} \leq C \int \frac{|f|}{\lambda} \left(1 + \ln^+ \frac{|f|}{\lambda}\right)^{\frac{q}{2}-1} d\mu. \quad (1.4)$$

In (1.3) and (1.4), q cannot be replaced by any smaller number.

Consider now the horizontal Littlewood-Paley-Stein functions and related maximal operators, defined for $f \in C_0^\infty$ by

$$\begin{aligned} h_k(f)(x) &= \left(\int_0^{+\infty} \left| t^k \frac{\partial^k}{\partial t^k} e^{t\Delta_v} f(x) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}, & H_k(f)(x) &= \sup_{t>0} \left| t^k \frac{\partial^k}{\partial t^k} e^{t\Delta_v} f(x) \right|, \\ g_k(f)(x) &= \left(\int_0^{+\infty} \left| t^k \frac{\partial^k}{\partial t^k} e^{-t\sqrt{-\Delta_v}} f(x) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}, & G_k(f)(x) &= \sup_{t>0} \left| t^k \frac{\partial^k}{\partial t^k} e^{-t\sqrt{-\Delta_v}} f(x) \right|. \end{aligned}$$

Here $k \geq 1$ for h_k and g_k , but $k \geq 0$ for H_k and G_k .

These operators are bounded on L^p for $1 < p < +\infty$ in the setting of a general symmetric semigroup with the contraction property, see [28]. The weak type $(1, 1)$ property of G_k , $k \geq 0$, in a general setting is obtained in [19]. For the other three operators, we have the following endpoint estimates. We remark that the weak type $(1, 1)$ of H_0 was obtained in [20, Theorem 2].

Theorem 3 *The operators h_1 , H_1 and g_k with $k \geq 1$ are of weak type $(1, 1)$. For $k \geq 2$, h_k and H_k are not of weak type $(1, 1)$; however, there exists a constant $C = C(v, k)$ such that for all $f \in L(1 + \ln^+ L)^{\frac{k}{2}-\frac{3}{4}}(\mu)$*

$$\mu \{x; h_k f(x) > \lambda\} \leq C \int \frac{|f|}{\lambda} \left(1 + \ln^+ \frac{|f|}{\lambda}\right)^{\frac{k}{2}-\frac{3}{4}} d\mu \quad \forall \lambda > 0, \quad (1.5)$$

and for all $f \in L(1 + \ln^+ L)^{\frac{k}{2}-\frac{1}{2}}(\mu)$

$$\mu \{x; H_k f(x) > \lambda\} \leq C \int \frac{|f|}{\lambda} \left(1 + \ln^+ \frac{|f|}{\lambda}\right)^{\frac{k}{2}-\frac{1}{2}} d\mu \quad \forall \lambda > 0. \quad (1.6)$$

In these two estimates, the exponents cannot be replaced by any smaller numbers.

Our estimates, in particular those involving Orlicz spaces, are optimal and go beyond earlier known results.

The structure of this paper is as follows. Section 2 contains estimates for the kernels of the Riesz transforms, which are used in Section 3 to prove Theorem 1. A fundamental tool here is Proposition 7 that is also needed in the later sections. The two parts of Theorem 2 are proved in Sections 4 and 5. Section 6 gives estimates for the time derivatives of the heat kernel, which are applied in the proof of Theorem 3 in Section 7.

1.1 Notation and simple facts

We assume that $v = e_1 = (1, 0, \dots, 0)$, which is no restriction, see [20]. Then $d\mu = e^{2x_1} dx$, and it will be convenient to write points in \mathbb{R}^n as $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$. Further, we denote $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^+ = \{1, 2, \dots\}$.

In the following, C denotes various constants which depend only on n and D in Sections 2–5, and only on n and k in Sections 6–7. By $A \lesssim B$, we mean $A \leq CB$ with such a C (we say that A is controlled by B), and $A \sim B$ stands for $A \leq CB$ and $B \leq CA$.

We have for any $x \in \mathbb{R}^n$ and $r \leq 1$,

$$\mu(B(x, r)) \sim r^n e^{2x_1}. \quad (1.7)$$

It follows that our space has the local doubling property,

The heat kernel $p_t(x, y)$ in our setting is defined through

$$e^{t\Delta_{e_1}} f(x) = \int p_t(x, y) f(y) d\mu(y), \quad t > 0, \quad x \in \mathbb{R}^n,$$

for suitable functions f . It is explicitly given by (cf. [16] p. 258)

$$p_t(x, y) = (4\pi t)^{-\frac{n}{2}} e^{-x_1 - y_1} e^{-t} e^{-\frac{|x-y|^2}{4t}} \quad (1.8)$$

$$= (4\pi t)^{-\frac{n}{2}} e^{-x_1 - y_1 - |x-y|} e^{-t\left(\frac{|x-y|}{2t} - 1\right)^2}, \quad (1.9)$$

for all $t > 0$ and $x = (x_1, x')$, $y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{n-1}$.

Our results include the simpler case $n = 1$, but the proofs are written for $n \geq 2$. Except for Proposition 7, we leave it to the reader to see how the arguments simplify for $n = 1$.

2 Estimates of the Riesz kernels

With D , k and q as in Theorem 1, we write $R_D(x, y)$ for the kernel of the Riesz transform $R_D = D(-\Delta_{e_1})^{-k/2}$. The multiindex α will be split as $\alpha = (\alpha_1, \alpha')$.

We state and prove local and global estimates for $R_D(x, y)$.

Proposition 4 *The kernel $R_D(x, y)$ satisfies the local standard estimates*

$$|R_D(x, y)| \lesssim \frac{1}{\mu(B(x, |x-y|))}, \quad 0 < |x-y| \leq 1. \quad (2.1)$$

and

$$|\nabla_y R_D(x, y)| \lesssim \frac{1}{|x-y| \mu(B(x, |x-y|))}, \quad 0 < |x-y| \leq 1. \quad (2.2)$$

Proposition 5 *The kernel $R_D(x, y)$ satisfies*

$$|R_D(x, y)| \lesssim e^{-x_1 - y_1 - |x - y|} |x - y|^{\frac{q-n-1}{2}} \left[1 + \left(\frac{|x' - y'|^2}{|x - y|} \right)^{\frac{k}{2}} \right], \quad |x - y| > 1. \quad (2.3)$$

This estimate is sharp in the sense that there exists a ball $B \subset \mathbb{R}^{n-1}$, depending only on n and D , such that

$$(-1)^k R_D(x, y) \sim e^{-x_1 - y_1 - |x - y|} (x_1 - y_1)^{\frac{q-n-1}{2}} \quad \text{for} \quad \frac{x' - y'}{(x_1 - y_1)^{1/2}} \in B, \quad (2.4)$$

if $x_1 - y_1 > 0$ is large enough.

We will need the following integral estimate.

Lemma 6 *With $\nu \in \mathbb{R}$ and $a > 0$, let*

$$B_\nu(a) = \int_0^{+\infty} t^\nu e^{-t} e^{-\frac{a^2}{4t}} \frac{dt}{t}.$$

(a) Then for $a \leq 2$

$$B_\nu(a) \sim \begin{cases} a^{2\nu} & \text{if } \nu < 0; \\ \log(1 + a^{-1}) & \text{if } \nu = 0; \\ 1 & \text{if } \nu > 0. \end{cases}$$

(b) Moreover,

$$B_\nu(a) = \sqrt{2\pi} 2^{-\nu} a^{\nu-1/2} e^{-a} (1 + O(a^{-1})), \quad a \rightarrow \infty,$$

and this estimate remains true if $B_\nu(a)$ is replaced by

$$\int_{(a/2-\ell) \vee 0}^{a/2+\ell} t^\nu e^{-t} e^{-\frac{a^2}{4t}} \frac{dt}{t}.$$

for any $\ell \geq a^{3/4}$.

In this lemma, the implicit constants depend on ν but not on a .

Lemma 6 can be proved either with elementary estimates or by evaluating the integral in terms of Bessel functions according to the 12th equality of §3.471 in [15]. The latter option also requires an easy estimate of those parts of the integral left out in the last statement.

We will also use a trivial extension of this lemma. Let $Q(t) = \sum_\nu q_\nu t^\nu$ be any finite linear combination of real powers of the variable $t > 0$. Then Lemma 6 (b) implies that

$$\int_0^{+\infty} Q(t) e^{-t} e^{-\frac{a^2}{4t}} \frac{dt}{t} = \sqrt{2\pi} Q(a/2) a^{-1/2} e^{-a} (1 + O(a^{-1})), \quad a \rightarrow \infty. \quad (2.5)$$

To prepare for the proof of the two propositions, we deduce an expression for $R_D(x, y)$. From (1.8) it follows that the kernel of $(-\Delta_{e_1})^{-k/2}$ is

$$\begin{aligned} (-\Delta_{e_1})^{-\frac{k}{2}}(x, y) &= \frac{1}{\Gamma(k/2)} \int_0^{+\infty} t^{\frac{k}{2}} p_t(x, y) \frac{dt}{t} \\ &= (4\pi)^{-\frac{n}{2}} \frac{1}{\Gamma(k/2)} e^{-x_1-y_1} \int_0^{+\infty} t^{\frac{k-n}{2}} e^{-t} e^{-\frac{|x-y|^2}{4t}} \frac{dt}{t}. \end{aligned}$$

Further,

$$R_D(x, y) = D_x(-\Delta_{e_1})^{-\frac{k}{2}}(x, y) = \frac{1}{\Gamma(k/2)} \int_0^{+\infty} t^{\frac{k}{2}} D_x p_t(x, y) \frac{dt}{t}. \quad (2.6)$$

Recall that the Hermite polynomial of degree $j \in \mathbb{N}$ is defined by

$$H_j(s) = (-1)^j e^{s^2} \frac{d^j}{ds^j} e^{-s^2},$$

and $H_\alpha = \otimes_i H_{\alpha_i}$ for any multiindex α . It is well known that the H_α are orthogonal with respect to the Gaussian measure $e^{-|x|^2} dx$ in \mathbb{R}^n and that the leading term of $H_j(s)$ is $2^j s^j$.

The definition of H_α implies that

$$\partial_x^\alpha e^{-\frac{|x-y|^2}{4t}} = \left(-\frac{1}{2\sqrt{t}} \right)^{|\alpha|} H_\alpha \left(\frac{x-y}{2\sqrt{t}} \right) e^{-\frac{|x-y|^2}{4t}}. \quad (2.7)$$

The kernel $R_D(x, y)$ is a linear combination of terms $\partial_x^\alpha (-\Delta_{e_1})^{-k/2}(x, y)$, with $\alpha = (\alpha_1, \alpha')$ a multiindex of length $|\alpha| = k$ and with $\alpha_1 \leq q$. When we differentiate the expression in (1.8), some differentiations with respect to x_1 may fall on the factor e^{-x_1} . Those differentiations falling on the factor $e^{-\frac{|x-y|^2}{4t}}$ will be given by a multiindex $\tilde{\alpha} = (\tilde{\alpha}_1, \alpha')$, with $0 \leq \tilde{\alpha}_1 \leq \alpha_1$. We then see from (2.7), with α replaced by $\tilde{\alpha}$, that $\partial_x^\alpha (-\Delta_{e_1})^{-k/2}(x, y)$ will be a sum of terms

$$\begin{aligned} &(-1)^{\alpha_1-\tilde{\alpha}_1} e^{-x_1-y_1} \int_0^{+\infty} t^{\frac{k-n}{2}} e^{-t} \left(-\frac{1}{2\sqrt{t}} \right)^{|\tilde{\alpha}|} H_{\tilde{\alpha}} \left(\frac{x-y}{2\sqrt{t}} \right) e^{-\frac{|x-y|^2}{4t}} \frac{dt}{t} \\ &= (-1)^{|\alpha|} 2^{-|\tilde{\alpha}|} e^{-x_1-y_1} \int_0^{+\infty} t^{\frac{\alpha_1-\tilde{\alpha}_1-n}{2}} H_{\tilde{\alpha}_1} \left(\frac{x_1-y_1}{2\sqrt{t}} \right) H_{\alpha'} \left(\frac{x'-y'}{2\sqrt{t}} \right) e^{-t} e^{-\frac{|x-y|^2}{4t}} \frac{dt}{t}, \end{aligned} \quad (2.8)$$

where $\tilde{\alpha}_1$ runs from 0 to α_1 and the terms have positive coefficients coming from Leibniz' formula and from the factors $(4\pi)^{-n/2}$ and $1/\Gamma(k/2)$.

Expanding the Hermite polynomials here, we obtain a sum of terms proportional to

$$e^{-x_1-y_1} (x_1-y_1)^{\gamma_1} (x'-y')^{\gamma'} \int_0^{+\infty} t^{\frac{\alpha_1-\tilde{\alpha}_1-\gamma_1-|\gamma'|-n}{2}} e^{-t} e^{-\frac{|x-y|^2}{4t}} \frac{dt}{t}, \quad (2.9)$$

and the sum is now taken also over a multiindex (γ_1, γ') with $0 \leq \gamma_1 \leq \tilde{\alpha}_1$ and $0 \leq \gamma' \leq \alpha'$ (componentwise ordering).

Proof of Proposition 4. Here $|x - y| \leq 1$. If the exponent of t in the integral in (2.9) is negative, we see from Lemma 6 (a) that the modulus of the expression (2.9) is controlled by

$$e^{-2x_1} |x - y|^{\gamma_1 + |\gamma'|} |x - y|^{\alpha_1 - \tilde{\alpha}_1 - \gamma_1 - |\gamma'| - n} \leq e^{-2x_1} |x - y|^{-n}.$$

For other values of the exponent, the bound $e^{-2x_1} |x - y|^{-n}$ also follows. This is the first standard estimate (2.1). To obtain also (2.2), it is enough to trace the argument given, with a differentiation also in y . \square

Proof of Proposition 5. For $|x - y| > 1$, Lemma 6 (b) implies that the expression (2.9) equals constant times

$$e^{-x_1 - y_1 - |x - y|} (x_1 - y_1)^{\gamma_1} (x' - y')^{\gamma'} |x - y|^{\frac{\alpha_1 - \tilde{\alpha}_1 - \gamma_1 - |\gamma'| - n - 1}{2}} (1 + O(|x - y|^{-1})), \quad (2.10)$$

whose modulus is controlled by

$$e^{-x_1 - y_1 - |x - y|} |x - y|^{\frac{\alpha_1 - (\tilde{\alpha}_1 - \gamma_1) - n - 1}{2}} \left(\frac{|x' - y'|^2}{|x - y|} \right)^{\frac{|\gamma'|}{2}}. \quad (2.11)$$

Now (2.3) follows, because $\alpha_1 \leq q$ and $\gamma_1 \leq \tilde{\alpha}_1$ and also $0 \leq |\gamma'| \leq |\alpha'| = k - \alpha_1$.

To verify (2.4), assume that $x_1 - y_1 > 0$ is large and that $|x' - y'|^2/(x_1 - y_1)$ stays bounded. In this argument, we will neglect all terms which are much smaller than the right-hand side of (2.4). From (2.10) and (2.11), we then see that in (2.8) we need only take $\alpha_1 = q$, which implies $|\alpha'| = k - q$, and $\gamma_1 = \tilde{\alpha}_1$. The latter equality means that in the Hermite polynomial $H_{\tilde{\alpha}_1}$ in (2.8), we consider only the leading term, which is

$$2^{\tilde{\alpha}_1} \left(\frac{x_1 - y_1}{2\sqrt{t}} \right)^{\tilde{\alpha}_1} = t^{-\frac{\tilde{\alpha}_1}{2}} (x_1 - y_1)^{\tilde{\alpha}_1}.$$

Instead of (2.8) we will now have

$$(-1)^k 2^{-|\tilde{\alpha}|} e^{-x_1 - y_1} (x_1 - y_1)^{\tilde{\alpha}_1} \int_0^{+\infty} t^{\frac{q - 2\tilde{\alpha}_1 - n}{2}} H_{\alpha'} \left(\frac{x' - y'}{2\sqrt{t}} \right) e^{-t} e^{-\frac{|x - y|^2}{4t}} \frac{dt}{t}.$$

Applying (2.5) to the integral here, we see that (2.8) amounts to a positive constant times

$$\begin{aligned} (-1)^k 2^{-|\tilde{\alpha}|} e^{-x_1 - y_1 - |x - y|} (x_1 - y_1)^{\tilde{\alpha}_1} \left(\frac{|x - y|}{2} \right)^{\frac{q - 2\tilde{\alpha}_1 - n}{2}} |x - y|^{-\frac{1}{2}} H_{\alpha'} \left(\frac{x' - y'}{\sqrt{2} |x - y|^{1/2}} \right) \\ \times (1 + O(|x - y|^{-1})). \end{aligned} \quad (2.12)$$

Here we can replace the powers of $|x - y|$ by the same powers of $x_1 - y_1$, since $|x - y| = x_1 - y_1 + O(1)$, so (2.12) equals

$$(-1)^k 2^{-\frac{q}{2} + \tilde{\alpha}_1 - |\tilde{\alpha}| + \frac{n}{2}} e^{-x_1 - y_1 - |x - y|} (x_1 - y_1)^{\frac{q - n - 1}{2}} H_{\alpha'} \left(\frac{x' - y'}{\sqrt{2} (x_1 - y_1)^{1/2}} \right) (1 + O(|x - y|^{-1})).$$

Summing over $0 \leq \tilde{\alpha}_1 \leq q$, we conclude that

$$\begin{aligned} \partial_x^\alpha (-\Delta_{\epsilon_1})^{-k/2}(x, y) \\ = (-1)^k b_\alpha e^{-x_1-y_1-|x-y|} (x_1 - y_1)^{\frac{q-n-1}{2}} H_{\alpha'} \left(\frac{x' - y'}{\sqrt{2}(x_1 - y_1)^{1/2}} \right) (1 + O(|x - y|^{-1})) \end{aligned}$$

for some $b_\alpha > 0$. Here $\alpha = (q, \alpha')$ and $|\alpha| = k$, and if we sum over such α with the coefficients from (1.1), the result will be

$$\begin{aligned} R_D(x, y) \\ = (-1)^k e^{-x_1-y_1-|x-y|} (x_1 - y_1)^{\frac{q-n-1}{2}} \sum a_\alpha b_\alpha H_{\alpha'} \left(\frac{x' - y'}{\sqrt{2}(x_1 - y_1)^{1/2}} \right) + \text{negligible terms.} \end{aligned}$$

Since the a_α do not all vanish, the orthogonality property of the Hermite polynomials implies that the polynomial given by the sum here is not identically 0. To finish the proof of (2.4), we need only take a closed ball B where this polynomial does not vanish. \square

3 Proof of Theorem 1

We split R_D into a part at infinity

$$R_D^\infty f(x) = \int_{|x-y|>1} R_D(x, y) f(y) d\mu(y)$$

and a local part $R_D^{\text{loc}} = R_D - R_D^\infty$.

The local part is easy to treat. Because of the local doubling property, we can use the method of localization. In view of Proposition 4, standard Calderón-Zygmund singular integral theory gives the weak type $(1, 1)$ of R_D^{loc} .

To estimate R_D^∞ , we start with that part defined by the restriction $x_1 - y_1 \leq 1$. Then we have integrability, since (2.3) implies

$$\int_{\substack{|x-y|>1 \\ x_1-y_1 \leq 1}} |R_D(x, y)| d\mu(x) \lesssim \int_{|x-y|>1} e^{1-|x-y|} |x - y|^{\frac{q-n-1}{2}} \left[1 + \left(\frac{|x' - y'|^2}{|x - y|} \right)^{\frac{k}{2}} \right] dx \lesssim 1,$$

uniformly in y . It follows that this part of the operator is of strong type $(1, 1)$.

To treat the opposite case $x_1 - y_1 > 1$, we write the first exponent in (2.3) as

$$-2x_1 - (|x - y| - (x_1 - y_1))$$

and observe that in this case

$$|x - y| - (x_1 - y_1) = \frac{|x' - y'|^2}{|x - y| + x_1 - y_1}$$

and

$$\frac{|x' - y'|^2}{2|x - y|} \leq \frac{|x' - y'|^2}{|x - y| + x_1 - y_1} \leq \frac{|x' - y'|^2}{|x - y|}. \quad (3.1)$$

Thus Proposition 5 implies that for $x_1 - y_1 > 1$

$$|R_D(x, y)| \lesssim e^{-2x_1} |x - y|^{\frac{q-n-1}{2}} \exp\left(-\frac{|x' - y'|^2}{2|x - y|}\right) \left[1 + \left(\frac{|x' - y'|^2}{|x - y|}\right)^{\frac{k}{2}}\right].$$

It follows that this part of $|R_D(x, y)|$ is controlled by the kernel

$$\mathcal{V}_\kappa(x, y) = e^{-2x_1} |x - y|^{\frac{\kappa-n-1}{2}} \exp\left(-\frac{1}{4} \frac{|x' - y'|^2}{|x - y|}\right) \chi_{\{x_1 - y_1 > 1\}}, \quad (3.2)$$

with $\kappa = q$.

Let for $\kappa \in \mathbb{R}$ and suitable functions f

$$\widetilde{\mathcal{V}}_\kappa f(x) = \int_{\mathbb{R}^n} \mathcal{V}_\kappa(x, y) f(y) d\mu(y). \quad (3.3)$$

To prove Theorem 1 in the case $q \leq 2$, it is enough to show the following result.

Proposition 7 *The operator $\widetilde{\mathcal{V}}_2$ defined by (3.3) is of weak type $(1, 1)$.*

The case $n = 1$ is trivial, since then

$$|\widetilde{\mathcal{V}}_2 f(x)| \leq e^{-2x} \|f\|_{L^1(d\mu)},$$

which implies the weak type $(1, 1)$ of $\widetilde{\mathcal{V}}_2$.

In what follows, we suppose that $n \geq 2$, so that

$$\widetilde{\mathcal{V}}_2 f(x) = e^{-2x_1} \int_{y_1 < x_1 - 1} |x - y|^{\frac{1-n}{2}} \exp\left(-\frac{1}{4} \frac{|x' - y'|^2}{|x - y|}\right) f(y) e^{2y_1} dy,$$

where we may assume $0 \leq f \in L^1(d\mu)$. It will be convenient to use $g(y) = f(y) e^{2y_1} \in L^1(dy)$ instead of f . When $|x' - y'| \lesssim \sqrt{x_1 - y_1}$, the exponential in the integrand is essentially 1. This indicates the most important part of the operator, dealt with in the following proposition.

Proposition 8 *The operator*

$$Tg(x) = e^{-2x_1} \int_{\substack{y_1 < x_1 - 1 \\ |x' - y'| < \sqrt{x_1 - y_1}}} (x_1 - y_1)^{\frac{1-n}{2}} g(y) dy \quad (3.4)$$

maps $L^1(dy)$ boundedly into $L^{1,\infty}(d\mu)$.

We first verify that this proposition implies Proposition 7. Let for $j = 1, 2, \dots$

$$T_j g(x) = e^{-2x_1} \int_{2^{j-1} \sqrt{x_1 - y_1} \leq |x' - y'| < 2^j \sqrt{x_1 - y_1}}^{y_1 < x_1 - 1} (x_1 - y_1)^{\frac{1-n}{2}} \exp\left(-\frac{1}{4} \frac{|x' - y'|^2}{|x - y|}\right) g(y) dy;$$

then $\widetilde{\mathcal{V}}_2 f \leq Tg + \sum_1^\infty T_j g$. In this integral, the exponential is less than $\exp(-c2^j)$ for some $c > 0$, so that

$$T_j g(x) \leq \exp(-c2^j) e^{-2x_1} \int_{|x' - y'| < 2^j \sqrt{x_1 - y_1}}^{y_1 < x_1 - 1} (x_1 - y_1)^{\frac{1-n}{2}} g(y) dy.$$

Scaling the variables x' and y' by a factor 2^j , we can control this expression in terms of the operator T , still with a rapidly decreasing coefficient. One can then apply Proposition 8 and use [29, Lemma 2.3] to sum in weak L^1 . The result will be the weak type $(1, 1)$ of $\widetilde{\mathcal{V}}_2$, which is Proposition 7.

Proof of Proposition 8. We cover \mathbb{R}^{n-1} and \mathbb{R}^n with lattices of unit cubes

$$Q_m = \{x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1} : m_\nu - 1 < x_\nu \leq m_\nu, \nu = 2, \dots, n\},$$

where $m = (m_2, \dots, m_n) \in \mathbb{Z}^{n-1}$, and

$$Q_{i,m} = (i-1, i] \times Q_m, \quad (i, m) \in \mathbb{Z} \times \mathbb{Z}^{n-1}.$$

In this proof, the word “cube” will always refer to cubes of these lattices.

Given $0 \leq g \in L^1(dy)$ and $\lambda > 0$, we must estimate the μ measure of the level set $L_\lambda = \{x : Tg(x) > \lambda\}$. Since $Tg(x) \leq e^{-2x_1} \int g(y) dy$, the coordinate x_1 is bounded above on L_λ . Without restriction, we can then assume that L_λ is contained in the left half-plane $\{x : x_1 \leq 0\}$.

The aim of the following lemma is to discretize the variable x_1 .

Lemma 9 *Let $i \leq 0$ and assume that the cube $Q_{i,m}$ intersects the level set L_λ . Then $\tilde{T}g(i, x') > e^{-2} 2^{-(n-1)/2} \lambda$ for all $x' \in Q_m$, where*

$$\tilde{T}g(i, x') = e^{-2i} \int_{|y' - x'| < \sqrt{i - y_1} + \sqrt{n-1}}^{y_1 < i-1} (i - y_1)^{\frac{1-n}{2}} g(y) dy. \quad (3.5)$$

Proof. Take a point $z \in Q_{i,m} \cap L_\lambda$, so that

$$\lambda < Tg(z) = e^{-2z_1} \int_{|y' - z'| < \sqrt{z_1 - y_1}}^{y_1 < z_1 - 1} (z_1 - y_1)^{\frac{1-n}{2}} g(y) dy. \quad (3.6)$$

Here $i < 1 + z_1$ and $z_1 - y_1 > 1$, so that $i - y_1 < 1 + z_1 - y_1 < 2(z_1 - y_1)$ and thus $(z_1 - y_1)^{(1-n)/2} < 2^{(n-1)/2} (i - y_1)^{(1-n)/2}$. Further, the region of integration in (3.6) is contained in that in (3.5), because $z_1 \leq i$ and $|z' - x'| \leq \sqrt{n-1}$ for any $x' \in Q_m$. Since

also $e^{-2z_1} \leq e^2 e^{-2i}$, it follows that $Tg(z) < e^2 2^{(n-1)/2} \tilde{T}g(i, x')$, which proves the lemma. \square

The region of integration in (3.5) is contained in the set

$$D_{i,m} = \bigcup_{x' \in Q_m} \left\{ (y_1, y') : y_1 < i - 1, |y' - x'| < \sqrt{i - y_1} + \sqrt{n - 1} \right\}.$$

For each cube $Q_{i,m}$, we define a family of “forbidden” cubes

$$F_{i,m} = \left\{ Q_{i',m'} : i' < i, |m - m'| < 4\sqrt{i - i'} + 4\sqrt{n - 1} \right\}$$

whose union contains $D_{i,m}$, that is,

$$D_{i,m} \subset \hat{D}_{i,m} := \bigcup_{Q \in F_{i,m}} Q,$$

as easily verified. The exponential behavior of μ implies that

$$\mu(\hat{D}_{i,m}) \lesssim \mu(Q_{i,m}). \quad (3.7)$$

For each $i \leq 0$, we will select some of the cubes $Q_{i,m}$, namely those for which m is in a set $A_i \subset \mathbb{Z}^{n-1}$. These A_i will be defined by recursion. We will have the inclusion

$$L_\lambda \subset \bigcup_{i \leq 0} \bigcup_{m \in A_i} (Q_{i,m} \cup \hat{D}_{i,m}). \quad (3.8)$$

Because of (3.7), this implies that

$$\mu(L_\lambda) \lesssim \sum_{i \leq 0} \sum_{m \in A_i} \mu(Q_{i,m}) \sim \sum_{i \leq 0} \sum_{m \in A_i} e^{2i}. \quad (3.9)$$

When we make the selection, i.e., define the A_i , we will consider the $\hat{D}_{i,m}$ as forbidden regions, where no selection is allowed. Indeed, we will select for each $i = 0, -1, -2, \dots$ those cubes $Q_{i,m}$ which intersect the level set and are not in $F_{j,\ell}$ for any already selected cube $Q_{j,\ell}$. In this way, cubes selected at different steps will be far from each other.

More precisely, we first let

$$A_0 = \{m \in \mathbb{Z}^{n-1} : Q_{0,m} \cap L_\lambda \neq \emptyset\}.$$

Assuming for some $i < 0$ the sets A_{i+1}, \dots, A_0 already defined, we then let

$$A_i = \{m \in \mathbb{Z}^{n-1} : Q_{i,m} \cap L_\lambda \neq \emptyset \text{ and } Q_{i,m} \notin F_{j,\ell} \text{ for all } j > i, \ell \in A_j\}.$$

The inclusion (3.8) is immediate from this construction.

For each $i \leq 0$, we define

$$D_i = \bigcup_{m \in A_i} D_{i,m},$$

which is a union of cubes contained in $\{y : y_1 \leq i - 1\}$. In order to replace the D_i by pairwise disjoint sets, we let $E_0 = D_0$ and

$$E_i = D_i \setminus \bigcup_{j=i+1}^0 D_j, \quad i = -1, -2, \dots$$

Then we let $g_i = g \chi_{E_i}$ and observe that $g = \sum_{j=i}^0 g_j$ in D_i . This together with Lemma 9 implies that, for $i \leq 0$ and $x' \in Q_m$ with $m \in A_i$,

$$\lambda \lesssim \tilde{T}g(i, x') = e^{-2i} \sum_{j=i}^0 \int_{y_1 < i-1} (i - y_1)^{\frac{1-n}{2}} \int_{|y' - x'| < \sqrt{i - y_1} + \sqrt{n-1}} g_j(y) dy' dy_1. \quad (3.10)$$

The integral in y' in this expression can be seen as a convolution in \mathbb{R}^{n-1} . That leads to an estimate which is insufficient for our purpose, essentially because several values of i are used for the same g_j . To do better, we will replace i by j in the iterated integral in (3.10). Thus we first claim that

$$(i - y_1)^{\frac{1-n}{2}} g_j(y) \lesssim (j - y_1)^{\frac{1-n}{2}} g_j(y) \quad (3.11)$$

if y, x', m, i and j are as in (3.10). Then we observe that an immediate consequence of (3.10) and (3.11) is that if $x' \in Q_m$ for some $m \in A_i$, then

$$\lambda \lesssim e^{-2i} \sum_{j=i}^0 \int_{y_1 < j-1} (j - y_1)^{\frac{1-n}{2}} \int_{|y' - x'| < \sqrt{j - y_1} + \sqrt{n-1}} g_j(y) dy' dy_1, \quad (3.12)$$

where we also made the regions of integration larger.

Before we use (3.12) to finish the proof of the lemma, we verify the claim (3.11). Then we can obviously assume that $g_j(y) \neq 0$, which implies $y \in E_j \subset D_j$. The definition of D_j says that $y_1 < j - 1$ and $|y' - z'| < \sqrt{j - y_1} + \sqrt{n - 1}$ for some $z' \in Q_\ell$ and some $\ell \in A_j$. Clearly $|z' - \ell| \leq \sqrt{n - 1}$, so the triangle inequality leads to

$$|y' - \ell| < \sqrt{j - y_1} + 2\sqrt{n - 1}. \quad (3.13)$$

The inner integral in (3.10) is taken over those y' satisfying

$$|y' - x'| < \sqrt{i - y_1} + \sqrt{n - 1}. \quad (3.14)$$

Since also $|x' - m| \leq \sqrt{n - 1}$, (3.13) and (3.14) imply

$$|m - \ell| < \sqrt{j - y_1} + \sqrt{i - y_1} + 4\sqrt{n - 1} \leq 2\sqrt{j - y_1} + 4\sqrt{n - 1}. \quad (3.15)$$

On the other hand, the cube $Q_{i,m}$ was selected and thus not forbidden by the selected cube $Q_{j,\ell}$, i.e., $Q_{i,m} \notin F_{j,\ell}$. This means that $|m - \ell| \geq 4\sqrt{j - y_1} + 4\sqrt{n - 1}$. Combining

this inequality with (3.15), we get $4\sqrt{j-i} < 2\sqrt{j-y_1}$, so that $j-i < (j-y_1)/4$. Then we can write

$$j-y_1 = j-i+i-y_1 < \frac{1}{4}(j-y_1) + i-y_1$$

and thus

$$j-y_1 < \frac{4}{3}(i-y_1).$$

This implies (3.11), and (3.12) also follows.

We now integrate the inequality (3.12) in x' over the unit cube Q_m , to conclude that

$$e^{2i} \lesssim \frac{1}{\lambda} \sum_{j=i}^0 \int_{y_1 < j-1} \int_{Q_m} (j-y_1)^{\frac{1-n}{2}} \int_{|y'-x'| < \sqrt{j-y_1} + \sqrt{n-1}} g_j(y) dy' dx' dy_1. \quad (3.16)$$

The inner part of this expression,

$$G_j(y_1, x') := (j-y_1)^{\frac{1-n}{2}} \int_{|y'-x'| < \sqrt{j-y_1} + \sqrt{n-1}} g_j(y_1, y') dy',$$

is the value at x' of the convolution in \mathbb{R}^{n-1} of $g_j(y_1, \cdot)$ and the characteristic function of the ball $\{y' \in \mathbb{R}^{n-1} : |y'| < \sqrt{j-y_1} + \sqrt{n-1}\}$, essentially normalized in $L^1(\mathbb{R}^{n-1})$ by the first factor. Thus

$$\int_{\mathbb{R}^{n-1}} G_j(y_1, x') dx' \lesssim \int_{\mathbb{R}^{n-1}} g_j(y_1, y') dy', \quad (3.17)$$

uniformly in $j-y_1 > 1$.

Combining (3.16) with (3.9), we have

$$\begin{aligned} \mu(L_\lambda) &\lesssim \frac{1}{\lambda} \sum_{i \leq 0} \sum_{m \in A_i} \sum_{j=i}^0 \int_{y_1 < j-1} \int_{Q_m} G_j(y_1, x') dx' dy_1 \\ &= \frac{1}{\lambda} \sum_{j \leq 0} \int_{y_1 < j-1} \sum_{i=-\infty}^j \sum_{m \in A_i} \int_{Q_m} G_j(y_1, x') dx' dy_1. \end{aligned}$$

In the last expression here, we have a double sum over all $m \in A_i$ and $i \leq j$. The corresponding family of cubes Q_m is pairwise disjoint, since $m \in A_i$ implies $Q_{i',m} \in F_{i,m}$ for $i' < i$ by the definition of $F_{i,m}$. Thus the double sum of integrals over Q_m is no larger than the left-hand side of (3.17). As a result,

$$\mu(L_\lambda) \lesssim \frac{1}{\lambda} \sum_{j \leq 0} \int_{y_1 < j-1} \int_{\mathbb{R}^{n-1}} g_j(y_1, y') dy' dy_1 \leq \frac{1}{\lambda} \sum_{j \leq 0} \int_{\mathbb{R}^n} g_j(y) dy \leq \frac{1}{\lambda} \int g(y) dy.$$

This ends the proof of Proposition 8. \square

Consider now the case $q > 2$ in Theorem 1. The main part of the kernel $R_D(x, y)$ is controlled by the kernel \mathcal{V}_q defined in (3.2). Therefore, the estimate (1.2) is a consequence of the following proposition.

Proposition 10 *For any $\kappa > 2$, there exists a constant $C = C(\kappa, n) > 0$ such that for all $f \in L(1 + \ln^+ L)^{\kappa/2-1}(\mu)$ and $\lambda > 0$, we have*

$$\mu \left\{ x; \left| \widetilde{\mathcal{V}}_\kappa f(x) \right| > \lambda \right\} \leq C \int \frac{|f|}{\lambda} \left(1 + \ln^+ \frac{|f|}{\lambda} \right)^{\frac{\kappa}{2}-1} d\mu, \quad (3.18)$$

where the operator $\widetilde{\mathcal{V}}_\kappa$ is defined by (3.3).

Proof. We have for $h \in L^1(\mu)$

$$\widetilde{\mathcal{V}}_\kappa h(x) = \int_{y_1 < x_1 - 1} e^{-2(x_1 - y_1)} |x - y|^{\frac{\kappa - n - 1}{2}} \exp \left(-\frac{1}{4} \frac{|x' - y'|^2}{|x - y|} \right) h(y) dy.$$

Here we assume $h \geq 0$ and apply the elementary inequality

$$ab^{\frac{\kappa}{2}-1} \leq C_0 \left[a(1 + \ln^+ a)^{\frac{\kappa}{2}-1} + e^{\frac{b}{8}} \right], \quad a, b > 0,$$

easily proved by separating the cases $b \leq 10(1 + \ln^+ a)$ and $b > 10(1 + \ln^+ a)$. Here C_0 depends only on κ . Letting $a = h(y)$ and $b = |x - y|$, we get

$$\begin{aligned} \widetilde{\mathcal{V}}_\kappa h(x) &\leq C_0 \widetilde{\mathcal{V}}_2 \left(h(1 + \ln^+ h)^{\frac{\kappa}{2}-1} \right) (x) \\ &\quad + C_0 \int_{y_1 < x_1 - 1} e^{-2(x_1 - y_1)} |x - y|^{\frac{1-n}{2}} \exp \left(-\frac{1}{4} \frac{|x' - y'|^2}{|x - y|} \right) e^{\frac{|x-y|}{8}} dy. \end{aligned}$$

The integral here equals

$$\int_{z_1 > 1} e^{-2z_1} |z|^{\frac{1-n}{2}} \exp \left(-\frac{1}{4} \frac{|z'|^2}{|z|} \right) e^{\frac{|z|}{8}} dz,$$

and to see that it is finite, we split it in two:

$$\int_{\substack{z_1 > 1 \\ |z'| < 2z_1}} + \int_{\substack{z_1 > 1 \\ |z'| > 2z_1}} = I_1 + I_2.$$

In I_1 , we can estimate the second and third exponential factors by 1 and $e^{3z_1/8}$, respectively, and integrate first in z' . In I_2 , we have $|z| < 3|z'|/2$ and thus $|z'|^2/|z| > 2|z'|/3$, so that the second exponential in I_2 is less than $\exp(-|z'|/6)$. It is then enough to estimate the third exponential by $\exp(z_1/8 + |z'|/8)$.

It now follows that

$$\widetilde{\mathcal{V}}_\kappa h(x) \leq C_0 \widetilde{\mathcal{V}}_2 \left(h(1 + \ln^+ h)^{\frac{\kappa-2}{2}} \right) (x) + C_1$$

for some constant C_1 .

Given $f \in L(1 + \ln^+ L)^{(\kappa-2)/2}(\mu)$ and $\lambda > 0$, we let $h = 2C_1\lambda^{-1}|f|$. Then $|\widetilde{\mathcal{V}}_\kappa f(x)| > \lambda$ implies $\widetilde{\mathcal{V}}_\kappa(h)(x) > 2C_1$ and $\widetilde{\mathcal{V}}_2(h(1 + \ln^+ h)^{(\kappa-2)/2})(x) > C_1/C_0$. Since $\widetilde{\mathcal{V}}_2$ is of weak type $(1, 1)$ by Proposition 7, we conclude that

$$\mu\{|\widetilde{\mathcal{V}}_\kappa f(x)| > \lambda\} \lesssim \|h(1 + \ln^+ h)^{\frac{\kappa}{2}-1}\|_{L^1(\mu)} \sim \int \frac{|f|}{\lambda} \left(1 + \ln^+ \frac{|f|}{\lambda}\right)^{\frac{\kappa}{2}-1} d\mu.$$

Proposition 10 is proved. \square

To show that (1.2) is sharp, we will use (2.4).

Let $f = \chi_{B(0,1)}$. With $\eta > 0$ large, we consider $R_D f(x)$ in the region

$$\Omega_\eta = \left\{x : \eta < x_1 < \eta + 1, \quad \frac{x'}{\sqrt{\eta}} \in \frac{1}{2}B\right\}. \quad (3.19)$$

Here $\frac{1}{2}B$ denotes the concentric scaling of the ball B from (2.4), by a factor $1/2$. Let $x \in \Omega_\eta$ and $y \in B(0, 1)$. Then

$$\frac{x' - y'}{(x_1 - y_1)^{1/2}} = \frac{x'}{\sqrt{\eta}} + O\left(\frac{1}{\sqrt{\eta}}\right), \quad \eta \rightarrow \infty.$$

This implies that $(x' - y')/(x_1 - y_1)^{1/2} \in B$ for large η , and (2.4) applies. Further, (3.1) shows that $e^{-x_1 - y_1 - |x - y|} \sim e^{-2x_1}$, and so

$$(-1)^k R_D f(x) \sim e^{-2\eta} \eta^{\frac{q-n-1}{2}} = \lambda$$

for $x \in \Omega_\eta$, where the equality defines λ . Then $\eta \sim \ln \lambda^{-1}$, and

$$\mu(\Omega_\eta) \sim e^{2\eta} \eta^{\frac{n-1}{2}} = \lambda^{-1} \eta^{\frac{q-n-1}{2}} \eta^{\frac{n-1}{2}} \sim \lambda^{-1} (\ln \lambda^{-1})^{\frac{q}{2}-1}.$$

For $q > 2$ this shows that (1.2) is sharp, and in particular that R_D is not of weak type $(1, 1)$.

Theorem 1 is completely proved. \square

4 Proof of Theorem 2 (a)

Before proving this theorem, we find local and global estimates for $t^{k/2} D_x p_t(x, y)$ in the space $L^2(dt/t)$. They will be analogous to Propositions 4 and 5, with similar proofs.

Proposition 11 *For $0 < |x - y| \leq 1$, one has*

$$\|t^{\frac{k}{2}} D_x p_t(x, y)\|_{L^2(dt/t)} \lesssim \frac{1}{\mu(B(x, |x - y|))} \quad (4.1)$$

and

$$\|t^{\frac{k}{2}} \nabla_y D_x p_t(x, y)\|_{L^2(dt/t)} \lesssim \frac{1}{|x - y| \mu(B(x, |x - y|))}. \quad (4.2)$$

Proposition 12 For $|x - y| > 1$, one has

$$\|t^{\frac{k}{2}} D_x p_t(x, y)\|_{L^2(dt/t)} \lesssim e^{-x_1 - y_1 - |x - y|} |x - y|^{\frac{q-n}{2} - \frac{1}{4}} \left[1 + \left(\frac{|x' - y'|^2}{|x - y|} \right)^{\frac{k}{2}} \right]. \quad (4.3)$$

Proof of Propositions 11 and 12. Clearly,

$$\left| t^{\frac{k}{2}} D_x p_t(x, y) \right|^2 = \sum_{|\alpha| = |\beta| = k} a_\alpha \overline{a_\beta} t^k \partial_x^\alpha p_t(x, y) \partial_x^\beta p_t(x, y).$$

We write $\alpha = (\alpha_1, \alpha')$ as before and similarly $\beta = (\beta_1, \beta')$, assuming always $|\alpha| = |\beta| = k$. Using the expression (1.8) for $p_t(x, y)$ and (2.7), we see that $t^k \partial_x^\alpha p_t(x, y) \partial_x^\beta p_t(x, y)$ is the sum of positive factors times

$$e^{-2x_1 - 2y_1} t^{k - \frac{|\tilde{\alpha}| + |\tilde{\beta}|}{2} - n} H_{\tilde{\alpha}} \left(\frac{x - y}{2\sqrt{t}} \right) H_{\tilde{\beta}} \left(\frac{x - y}{2\sqrt{t}} \right) e^{-2t} e^{-\frac{|x - y|^2}{2t}},$$

taken over $\tilde{\alpha}_1 = 0, \dots, \alpha_1$ and $\tilde{\beta}_1 = 0, \dots, \beta_1$. Here as in Section 2, we have $\tilde{\alpha} = (\tilde{\alpha}_1, \alpha')$, and similarly $\tilde{\beta} = (\tilde{\beta}_1, \beta')$.

Integrating this expression with respect to dt/t , we obtain a positive constant times

$$e^{-2x_1 - 2y_1} \int_0^{+\infty} t^{k - \frac{|\tilde{\alpha}| + |\tilde{\beta}|}{2} - n} H_{\tilde{\alpha}} \left(\frac{x - y}{2\sqrt{t}} \right) H_{\tilde{\beta}} \left(\frac{x - y}{2\sqrt{t}} \right) e^{-2t} e^{-\frac{|x - y|^2}{2t}} \frac{dt}{t}. \quad (4.4)$$

Here the product of the two Hermite polynomials is a linear combination of terms

$$\left(\frac{x_1 - y_1}{2\sqrt{t}} \right)^{\gamma_1} \left(\frac{x' - y'}{2\sqrt{t}} \right)^{\gamma'} \left(\frac{x_1 - y_1}{2\sqrt{t}} \right)^{\delta_1} \left(\frac{x' - y'}{2\sqrt{t}} \right)^{\delta'},$$

where $\gamma = (\gamma_1, \gamma')$ and $\delta = (\delta_1, \delta')$ satisfy $0 \leq \gamma_1 \leq \tilde{\alpha}_1$ and $0 \leq \gamma' \leq \alpha'$, and similarly $0 \leq \delta_1 \leq \tilde{\beta}_1$ and $0 \leq \delta' \leq \beta'$. Each such term leads to a corresponding term in the expression (4.4), proportional to

$$e^{-2x_1 - 2y_1} (x_1 - y_1)^{\gamma_1 + \delta_1} (x' - y')^{\gamma' + \delta'} \int_0^{+\infty} t^{k - \frac{|\tilde{\alpha}| + |\tilde{\beta}| + |\gamma| + |\delta|}{2} - n} e^{-2t} e^{-\frac{|x - y|^2}{2t}} \frac{dt}{t}. \quad (4.5)$$

After replacing t by $\tau = 2t$ in this integral, one can apply Lemma 6 with $a = 2|x - y|$.

For $|x - y| \leq 1$, we see from Lemma 6 (a) that the modulus of (4.5) is controlled by

$$e^{-2x_1 - 2y_1} |x_1 - y_1|^{\gamma_1 + \delta_1} |x' - y'|^{|\gamma'| + |\delta'|} |x - y|^{2k - (|\tilde{\alpha}| + |\tilde{\beta}| + |\gamma| + |\delta|) - 2n} \lesssim e^{-4x_1} |x - y|^{-2n},$$

if the exponent of t in the integral is negative. But in the opposite case, this is also true. Thus (4.1) is proved. To complete the proof of Proposition 11, one verifies (4.2) in a similar way.

Aiming at Proposition 12, we assume that $|x - y| > 1$ and conclude from Lemma 6 (b) that (4.5) is in modulus bounded by constant times

$$e^{-2x_1-2y_1-2|x-y|} |x_1 - y_1|^{\gamma_1+\delta_1} |x' - y'|^{|\gamma'|+|\delta'|} |x - y|^{k-\frac{|\tilde{\alpha}|+|\tilde{\beta}|+|\gamma|+|\delta|}{2}-n-\frac{1}{2}}.$$

In this expression, we estimate $|x_1 - y_1|$ by $|x - y|$ and write k as $k = (\alpha_1 + |\alpha'| + \beta_1 + |\beta'|)/2$, getting

$$e^{-2x_1-2y_1-2|x-y|} |x - y|^{\frac{\alpha_1+\beta_1}{2}-\frac{\tilde{\alpha}_1-\gamma_1}{2}-\frac{\tilde{\beta}_1-\delta_1}{2}-n-\frac{1}{2}} \left(\frac{|x' - y'|}{|x - y|^{1/2}} \right)^{|\gamma'|+|\delta'|}. \quad (4.6)$$

Here we have

$$\alpha_1 \leq q, \quad \beta_1 \leq q, \quad \tilde{\alpha}_1 - \gamma_1 \geq 0, \quad \tilde{\beta}_1 - \delta_1 \geq 0 \quad (4.7)$$

and also $|\gamma'|, |\delta'| \leq k$. Thus for $|x - y| \geq 1$ the last expression is at most

$$e^{-2x_1-2y_1-2|x-y|} |x - y|^{q-n-\frac{1}{2}} \left[1 + \left(\frac{|x' - y'|^2}{|x - y|} \right)^k \right]. \quad (4.8)$$

We have shown that this is a bound for $\int_0^{+\infty} t^k |D_x p_t(x, y)|^2 dt/t$, and (4.3) follows.

The two propositions are proved. \square

Remark The estimate at the end of the proof just given can be sharp only when all the inequalities (4.7) are equalities. Indeed, if any of the four inequalities is strict, one can estimate (4.6) by a modified quantity (4.8) in which $|x - y|$ has a smaller exponent.

Proof of Theorem 2 (a). Clearly,

$$\mathcal{H}_D(f)(x) = \left(\int_0^{+\infty} \left| t^{\frac{k}{2}} \int D_x p_t(x, y) f(y) d\mu(y) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

We now introduce a local part $\mathcal{H}_D^{\text{loc}}$ and a global part \mathcal{H}_D^∞ of this operator, defined by restricting the integration in y here to the region $|y - x| \leq 1$ and $|y - x| > 1$, respectively. Thus

$$\mathcal{H}_D f \leq \mathcal{H}_D^{\text{loc}} f + \mathcal{H}_D^\infty f.$$

To deal with the local part, we apply the method of localization as done for the Riesz transforms, but now using vector-valued singular integral theory and Proposition 11.

For \mathcal{H}_D^∞ , we start by applying Minkowski's integral inequality, getting

$$\mathcal{H}_D^\infty(f)(x) \leq \int_{|x-y|>1} |f(y)| \left(\int_0^{+\infty} \left| t^{\frac{k}{2}} D_x p_t(x, y) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} d\mu(y).$$

We can now follow the argument for R_D^∞ in the proof of Theorem 1, replacing (2.3) by (4.3). In particular, that part of \mathcal{H}_D^∞ given by $x_1 - y_1 \leq 1$ is seen to be of strong type $(1, 1)$. Moreover, the kernel of the remaining part is controlled by $\mathcal{V}_\kappa(x, y)$ with $\kappa = q + 1/2$.

This leads to the weak type (1, 1) estimate of Theorem 2 (a) and also to (1.3), in view of Propositions 7 and 10.

For the sharpness parts of Theorem 2 (a), we let $f = \chi_{B(0,1)}$. Let $|\alpha| = k$ and take points y and x with $|y| < 1$ and $x_1 > 0$ large but $x'/\sqrt{|x|}$ bounded, which implies $x_1 = |x| + O(1)$. We will then estimate $t^{k/2} \partial_x^\alpha p_t(x, y)$, and we need to be more precise than in the preceding argument. Because of (1.8) and (2.7), this quantity is the sum of

$$(-1)^k \binom{\alpha_1}{\tilde{\alpha}_1} 2^{-|\tilde{\alpha}|} e^{-x_1-y_1} t^{\frac{k-|\tilde{\alpha}|-n}{2}} H_{\tilde{\alpha}} \left(\frac{x-y}{2\sqrt{t}} \right) e^{-t} e^{-\frac{|x-y|^2}{4t}}, \quad (4.9)$$

taken over $\tilde{\alpha}_1 = 0, \dots, \alpha_1$. Here $\tilde{\alpha} = (\tilde{\alpha}_1, \alpha')$ as before. The remark after the end of the proof of Propositions 11 and 12 shows that we need only consider the case $\alpha_1 = q$, which implies $|\alpha'| = k - q$. For the same reason, we may replace the polynomial $H_{\tilde{\alpha}_1}((x_1 - y_1)/2\sqrt{t})$ by its leading term $2^{\tilde{\alpha}_1} ((x_1 - y_1)/2\sqrt{t})^{\tilde{\alpha}_1}$.

Then we can write

$$\begin{aligned} H_{\tilde{\alpha}} \left(\frac{x-y}{2\sqrt{t}} \right) &= 2^{\tilde{\alpha}_1} \left(\frac{x_1 - y_1}{2\sqrt{t}} \right)^{\tilde{\alpha}_1} H_{\alpha'} \left(\frac{x' - y'}{2\sqrt{t}} \right) + \dots \\ &= 2^{\tilde{\alpha}_1} t^{\frac{\tilde{\alpha}_1}{2}} \left(\frac{x_1 - y_1}{2t} \right)^{\tilde{\alpha}_1} H_{\alpha'} \left(\frac{x' - y'}{2\sqrt{t}} \right) + \dots, \end{aligned}$$

where the dots indicate negligible terms. Restricting t by $\left| t - \frac{|x|}{2} \right| < |x|^{3/4}$, we see that

$$\begin{aligned} H_{\tilde{\alpha}} \left(\frac{x-y}{2\sqrt{t}} \right) &= 2^{\tilde{\alpha}_1} t^{\frac{\tilde{\alpha}_1}{2}} \left(\frac{x_1}{|x|} + O(|x|^{-\frac{1}{4}}) \right)^{\tilde{\alpha}_1} H_{\alpha'} \left(\frac{x'}{\sqrt{2|x|}} + O(|x|^{-\frac{1}{4}}) \right) + \dots \\ &= 2^{\tilde{\alpha}_1} t^{\frac{\tilde{\alpha}_1}{2}} H_{\alpha'} \left(\frac{x'}{\sqrt{2|x|}} \right) + \dots \end{aligned}$$

We now insert this last expression for the Hermite polynomial in (4.9), and observe that $|\tilde{\alpha}| = \tilde{\alpha}_1 + |\alpha'| = \tilde{\alpha}_1 + k - q$. Then the only factor in (4.9) which depends on $\tilde{\alpha}_1$ will be the binomial coefficient. Since

$$\sum_{\tilde{\alpha}_1=0}^{\alpha_1} \binom{\alpha_1}{\tilde{\alpha}_1} = 2^{\alpha_1} = 2^q,$$

we get by summing (4.9) in $\tilde{\alpha}_1$

$$t^{\frac{k}{2}} \partial_x^\alpha p_t(x, y) = (-1)^k 2^{2q-k} e^{-x_1-y_1} t^{\frac{q-n}{2}} H_{\alpha'} \left(\frac{x'}{\sqrt{2|x|}} \right) e^{-t} e^{-\frac{|x-y|^2}{4t}} + \dots$$

The next step is to sum these expressions in α , with the coefficients from (1.1). The result is that for y , x and t as described above,

$$\begin{aligned} t^{\frac{k}{2}} D_x p_t(x, y) &= (-1)^k 2^{2q-k} e^{-x_1-y_1} t^{\frac{q-n}{2}} e^{-t} e^{-\frac{|x-y|^2}{4t}} \sum_{|\alpha'|=k-q} a_{(q,\alpha')} H_{\alpha'} \left(\frac{x'}{\sqrt{2|x|}} \right) + \text{harmless terms.} \end{aligned}$$

The sum here is $P\left(x'/\sqrt{2|x|}\right)$ for some nonzero polynomial P in $n-1$ variables. Thus we can find a ball $B \subset \mathbb{R}^{n-1}$ in which P does not vanish. Further,

$$e^{-\frac{|x-y|^2}{4t}} \sim e^{-\frac{|x|^2}{4t}},$$

since $|y| < 1$ and $t \sim |x|$. With $f = \chi_{B(0,1)}$, we then get by integrating in y

$$\left| t^{\frac{k}{2}} D e^{t\Delta_{e_1}} f(x) \right|^2 \sim e^{-2x_1} x_1^{q-n} e^{-2t} e^{-\frac{|x|^2}{2t}}$$

where as before $x_1 > 0$ is large, $x'/\sqrt{2|x|} \in B$ and $\left| t - \frac{|x|}{2} \right| < |x|^{3/4}$. Hence,

$$\int_0^{+\infty} \left| t^{\frac{k}{2}} D e^{t\Delta_{e_1}} f(x) \right|^2 \frac{dt}{t} \gtrsim e^{-2x_1} x_1^{q-n} \int_{\frac{|x|}{2}-|x|^{3/4}}^{\frac{|x|}{2}+|x|^{3/4}} e^{-2t} e^{-\frac{|x|^2}{2t}} \frac{dt}{t}. \quad (4.10)$$

After the change of variable $\tau = 2t$, we can apply the last part of Lemma 6 (b) to conclude that the right-hand side of (4.10) equals a positive constant times

$$e^{-2x_1} x_1^{q-n} e^{-2|x|} |x|^{-\frac{1}{2}} (1 + O(|x|^{-1})) = e^{-2x_1-2|x|} x_1^{q-n-\frac{1}{2}} (1 + O(|x|^{-1})). \quad (4.11)$$

We define Ω_η by (3.19), though B is not the same as in Section 3. If η is large, $x \in \Omega_\eta$ implies $x'/\sqrt{2|x|} \in B$. It now follows from (4.10) and (4.11) that for large η

$$\mathcal{H}_D^\infty f \gtrsim e^{-2\eta} \eta^{\frac{q-n}{2}-\frac{1}{4}}.$$

in Ω_η . This implies the “only if” part of Theorem 2 (a) and the sharpness of q ; cf. the very last part of Section 3. We leave it to the reader to check that all the neglected terms above can be disregarded.

Theorem 2 (a) is proved. \square

5 Proof of Theorem 2 (b)

To prove the claimed boundedness properties of \mathcal{G}_α , we copy the reasoning used for Theorems 1 and 2 (a). We must estimate the kernel $t^k D_x P_t(x, y)$ in the space $L^2(dt/t)$.

The subordination formula says that the Poisson kernel P_t is given by

$$P_t(x, y) = \frac{t}{2\sqrt{\pi}} \int_0^{+\infty} u^{-\frac{3}{2}} e^{-\frac{t^2}{4u}} p_u(x, y) du.$$

This will also hold with $P_t(x, y)$ and $p_u(x, y)$ replaced by their derivatives $D_x P_t(x, y)$ and $D_x p_u(x, y)$, respectively. Using Minkowski's integral inequality, we conclude that

$$\begin{aligned} \left(\int_0^{+\infty} |t^k D_x P_t(x, y)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} &\sim \left(\int_0^{+\infty} \left| t^{k+1} \int_0^{+\infty} u^{-\frac{3}{2}} e^{-\frac{t^2}{4u}} D_x p_u(x, y) du \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \\ &\leq \int_0^{+\infty} u^{-\frac{3}{2}} |D_x p_u(x, y)| \left(\int_0^{+\infty} \left(t^{k+1} e^{-\frac{t^2}{4u}} \right)^2 \frac{dt}{t} \right)^{\frac{1}{2}} du. \end{aligned}$$

The last inner integral here is seen to equal $2^k \Gamma(k+1)u^{k+1}$, and so

$$\left(\int_0^{+\infty} |t^k D_x P_t(x, y)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \lesssim \int_0^{+\infty} u^{\frac{k}{2}} |D_x p_u(x, y)| \frac{du}{u}. \quad (5.1)$$

The right-hand side in (5.1) is like the expression for $R_D(x, y)$ in (2.6), except for the modulus signs. In Section 2, the equality (2.6) was used to prove Propositions 4 and 5. Tracing those arguments, one sees that the inequality (5.1) is sufficient to imply estimates similar to (2.1) and (2.3) for $(\int_0^{+\infty} |t^k D_x P_t(x, y)|^2 dt/t)^{1/2}$. Taking a derivative also in y , one obtains the analog of (2.2). This leads to the weak type of \mathcal{G}_α for $q \leq 2$ and also to (1.4), as seen from the proof of Theorem 1 in Section 3.

It remains to verify the sharpness parts of Theorem 2 (b). Recall that $D = \sum_{|\alpha|=k} a_\alpha \partial^\alpha$. We will find an $f \in L^1(\mu)$ and a ball $B \subset \mathbb{R}^{n-1}$ such that if $x_1 > 0$ is large and $x'/\sqrt{x_1} \in B$, one has

$$\mathcal{G}_D f(x) \gtrsim e^{-2x_1} x_1^{\frac{q-n-1}{2}}. \quad (5.2)$$

From this estimate, the sharpness parts of Theorem 2 (b) will follow, cf. the last few lines of Section 3.

To estimate $D_x P_t(x, y)$, we consider

$$\partial_x^\alpha P_t(x, y) = \frac{t}{2\sqrt{\pi}} \int_0^{+\infty} u^{-\frac{3}{2}} e^{-\frac{t^2}{4u}} \partial_x^\alpha p_u(x, y) du,$$

with $|\alpha| = k$ and $\alpha_1 \leq q$.

From (1.8) we see that $\partial_x^\alpha P_t(x, y)$ is a sum of terms

$$(-1)^{\alpha_1 - \tilde{\alpha}_1} (4\pi)^{-\frac{n+1}{2}} \binom{\alpha_1}{\tilde{\alpha}_1} e^{-x_1 - y_1} t \int_0^{+\infty} u^{-\frac{1+n}{2}} e^{-\frac{t^2}{4u}} e^{-u} \partial_x^{\tilde{\alpha}} e^{-\frac{|x-y|^2}{4u}} \frac{du}{u},$$

where $\tilde{\alpha} = (\tilde{\alpha}_1, \alpha')$ as before, and the sum is taken over $\tilde{\alpha}_1 = 0, \dots, \alpha_1$. Using (2.7), we see that this expression equals

$$(-1)^k \pi^{-\frac{n+1}{2}} 2^{-|\tilde{\alpha}| - n - 1} \binom{\alpha_1}{\tilde{\alpha}_1} e^{-x_1 - y_1} t \int_0^{+\infty} u^{-\frac{1+|\tilde{\alpha}|+n}{2}} H_{\tilde{\alpha}} \left(\frac{x-y}{2\sqrt{u}} \right) e^{-u} e^{-\frac{t^2 + |x-y|^2}{4u}} \frac{du}{u}. \quad (5.3)$$

We now apply (2.5) to the integral here. It follows that for large $|x-y|$ the quantity (5.3) equals

$$\begin{aligned} & (-1)^k (2\pi)^{-\frac{n}{2}} 2^{-\frac{|\tilde{\alpha}|}{2}} \binom{\alpha_1}{\tilde{\alpha}_1} t (t^2 + |x-y|^2)^{-\frac{1+|\tilde{\alpha}|+n}{4} - \frac{1}{4}} e^{-x_1 - y_1 - \sqrt{t^2 + |x-y|^2}} \\ & \times H_{\tilde{\alpha}} \left(\frac{x-y}{\sqrt{2}(t^2 + |x-y|^2)^{1/4}} \right) \left(1 + O \left(\frac{1}{\sqrt{t^2 + |x-y|^2}} \right) \right). \end{aligned} \quad (5.4)$$

Again we consider points x for which $x_1 > 0$ is large and $x'/\sqrt{x_1}$ stays bounded. Further, we assume $|y| < 1$ and restrict t by $\sqrt{x_1} < t < 2\sqrt{x_1}$. Some Taylor expansions then lead to $|x - y| = x_1(1 + O(x_1^{-1}))$ and

$$\begin{aligned}\sqrt{t^2 + |x - y|^2} &= |x| \left(1 - \frac{x \cdot y}{|x|^2} + \frac{t^2}{2|x|^2} + O\left(\frac{1}{|x|^2}\right) \right) \\ &= |x| - y_1 + \frac{t^2}{2x_1} + O\left(\frac{1}{\sqrt{x_1}}\right) \\ &= x_1(1 + O(x_1^{-1})),\end{aligned}\tag{5.5}$$

as $x_1 \rightarrow \infty$.

Using (5.5), we conclude that the expression (5.4) now equals

$$\begin{aligned}(-1)^k (2\pi)^{-\frac{n}{2}} 2^{-\frac{|\tilde{\alpha}|}{2}} \begin{pmatrix} \alpha_1 \\ \tilde{\alpha}_1 \end{pmatrix} t x_1^{-\frac{|\tilde{\alpha}|+n+2}{2}} e^{-x_1-|x|-\frac{t^2}{2x_1}} \\ \times H_{\tilde{\alpha}_1} \left(\sqrt{\frac{x_1}{2}} \right) H_{\alpha'} \left(\frac{x'}{\sqrt{2x_1}} \right) \left(1 + O\left(\frac{1}{\sqrt{x_1}}\right) \right),\end{aligned}\tag{5.6}$$

as $x_1 \rightarrow \infty$. We can replace the Hermite polynomial $H_{\tilde{\alpha}_1}(\sqrt{x_1/2})$ here by its leading term, which is $2^{\tilde{\alpha}_1/2} x_1^{\tilde{\alpha}_1/2}$. Since $|\tilde{\alpha}| = \tilde{\alpha}_1 + |\alpha'|$, this will make the exponents of 2 and x_1 in (5.6) independent of $\tilde{\alpha}_1$. When we then sum in $\tilde{\alpha}_1 = 0, \dots, \alpha_1$, the binomial coefficients in (5.6) will sum up to 2^{α_1} , and as a result

$$\partial_x^\alpha P_t(x, y) = (-1)^k (2\pi)^{-\frac{n}{2}} 2^{\alpha_1 - \frac{|\alpha'|}{2}} t x_1^{-\frac{|\alpha'|+n+2}{2}} e^{-x_1-|x|-\frac{t^2}{2x_1}} H_{\alpha'} \left(\frac{x'}{\sqrt{2x_1}} \right) \left(1 + O\left(\frac{1}{\sqrt{x_1}}\right) \right).$$

Next, we integrate against $f(y) d\mu(y)$ with $f = \chi_{B(0,1)}/\mu(B(0,1))$. Summing also over α , we get

$$\begin{aligned}D_x P_t f(x) \\ = (-1)^k (2\pi)^{-\frac{n}{2}} t x_1^{-\frac{n+2}{2}} e^{-x_1-|x|-\frac{t^2}{2x_1}} \sum_{\alpha} 2^{\alpha_1 - \frac{|\alpha'|}{2}} a_{\alpha} x_1^{-\frac{|\alpha'|}{2}} H_{\alpha'} \left(\frac{x'}{\sqrt{2x_1}} \right) \left(1 + O\left(\frac{1}{\sqrt{x_1}}\right) \right).\end{aligned}$$

For large x_1 , the largest terms in the sum here are those where $|\alpha'|$ takes its minimal value $k - q$, so that $\alpha_1 = q$. We introduce the nonzero polynomial

$$P(z') = (-1)^k (2\pi)^{-\frac{n}{2}} 2^{\frac{3q-k}{2}} \sum_{|\alpha'|=k-q} a_{(q,\alpha')} H_{\alpha'}(z').$$

Since $x'/\sqrt{2x_1}$ stays bounded, it follows that

$$t^k D_x P_t f(x) = e^{-x_1-|x|-\frac{t^2}{2x_1}} t^{k+1} x_1^{-\frac{k-q+n+2}{2}} \left(P \left(\frac{x'}{\sqrt{2x_1}} \right) + O\left(\frac{1}{\sqrt{x_1}}\right) \right).$$

We can now integrate in t and conclude that

$$(\mathcal{G}_D f(x))^2 \gtrsim e^{-4x_1} \int_{\sqrt{x_1}}^{2\sqrt{x_1}} e^{-\frac{t^2}{x_1}} t^{2k+2} \frac{dt}{t} x_1^{-(k-q+n+2)} \left(P\left(\frac{x'}{\sqrt{2x_1}}\right)^2 + O\left(\frac{1}{\sqrt{x_1}}\right) \right).$$

The integral here is of order of magnitude x_1^{k+1} , and so

$$\mathcal{G}_D f(x) \gtrsim e^{-2x_1} x_1^{\frac{q-n-1}{2}} \left(\left| P\left(\frac{x'}{\sqrt{2x_1}}\right) \right| + O\left(\frac{1}{\sqrt{x_1}}\right) \right).$$

To obtain (5.2), it is now enough to let B be a closed ball in \mathbb{R}^{n-1} in which the polynomial P does not vanish.

This ends the proof of Theorem 2 (b). \square

6 Sharp estimates for $\frac{\partial^k}{\partial t^k} p_t$

This section is a preparation for the proof of Theorem 3.

We take k derivatives with respect to t of

$$p_t(x, y) = (4\pi)^{-\frac{n}{2}} e^{-x_1 - y_1} t^{-\frac{n}{2}} e^{-t - \frac{|x-y|^2}{4t}}. \quad (6.1)$$

Those derivatives which fall on the last exponential here will produce a factor $\frac{|x-y|^2}{4t^2} - 1$. The derivative

$$\frac{\partial}{\partial t} \left[\frac{|x-y|^2}{4t^2} - 1 \right] = -\frac{|x-y|^2}{2t^3}$$

will also appear. We see that

$$\frac{\partial^k}{\partial t^k} p_t(x, y) = q_k p_t(x, y), \quad (6.2)$$

where the factor q_k is given by

$$q_k = Q_k \left(\frac{|x-y|^2}{4t^2} - 1, \frac{1}{t}, \frac{|x-y|^2}{2t^3} \right)$$

for a polynomial Q_k in three variables, whose coefficients depend only on n and k .

To estimate q_k , we examine the terms of this polynomial. Consider a term obtained by letting exactly m of the k differentiations fall on the exponential $e^{-t - |x-y|^2/4t}$. If, moreover, the number of differentiations falling on a power of $\frac{|x-y|^2}{4t^2} - 1$ is p , the resulting term of Q_k will contain p factors $-\frac{|x-y|^2}{2t^3}$ and $m - p$ factors $\frac{|x-y|^2}{4t^2} - 1$. The remaining $k - m - p$

differentiations will produce factors $-t^{-1}$. We conclude that q_k is a sum of expressions of the form

$$C_{m,p} \left(\frac{|x-y|^2}{4t^2} - 1 \right)^{m-p} \left(-\frac{|x-y|^2}{2t^3} \right)^p (-t)^{-(k-m-p)} \quad (6.3)$$

with $C_{m,p} > 0$. Here $p+m \leq k$ and $p \leq m$; thus $2p \leq k$ so that $p \leq [\frac{k}{2}]$. This implies the upper estimate

$$\left| \frac{\partial^k}{\partial t^k} p_t(x, y) \right| \lesssim \sum \left| \frac{|x-y|^2}{4t^2} - 1 \right|^{m-p} |x-y|^{2p} t^{-k+m-2p} p_t(x, y), \quad (6.4)$$

the sum taken over $0 \leq p \leq [\frac{k}{2}]$ and $0 \leq m \leq k-p$.

We also need a lower estimate. Let $\eta > 0$ be large and define

$$\Sigma_\eta = \{x; \eta - 1 < x_1 < \eta \text{ and } \sqrt{\eta} < x_i < 2\sqrt{\eta}, \ i = 2, \dots, n\}.$$

Lemma 13 For $x \in \Sigma_\eta$, $|y| < 1$ and

$$\frac{\eta}{2} \left(1 - 2\frac{c_1}{\sqrt{\eta}} \right) < t < \frac{\eta}{2} \left(1 - \frac{c_1}{\sqrt{\eta}} \right)$$

with $c_1 = c_1(n, k)$ small enough and η large enough, one has

$$(-1)^{[\frac{k}{2}]} t^k \frac{\partial^k}{\partial t^k} p_t(x, y) \gtrsim e^{-2\eta} \eta^{\frac{k-n}{2}}. \quad (6.5)$$

Proof. Assuming x, y and t as in the lemma, we can write

$$t = \frac{\eta}{2} \left(1 - \frac{\zeta}{\sqrt{\eta}} \right),$$

where $c_1 < \zeta < 2c_1$.

Some simple computations will lead to $|x-y| = \eta(1 + O(\eta^{-1}))$ and

$$\frac{|x-y|^2}{4t^2} - 1 = 2 \frac{\zeta}{\sqrt{\eta}} \left(1 + O(\eta^{-\frac{1}{2}}) \right) \quad \text{and} \quad \frac{|x-y|^2}{2t^3} = \frac{4}{\eta} \left(1 + O(\eta^{-\frac{1}{2}}) \right),$$

as $\eta \rightarrow \infty$. The expression (6.3) thus equals

$$\begin{aligned} (-1)^{m-k} C_{m,p} \left(2\frac{\zeta}{\sqrt{\eta}} \right)^{m-p} \left(\frac{4}{\eta} \right)^p \left(\frac{\eta}{2} \right)^{-k+m+p} \left(1 + O(\eta^{-\frac{1}{2}}) \right) \\ = (-1)^{m-k} C_{m,p} 2^k \eta^{\frac{m+p}{2}-k} \zeta^{m-p} \left(1 + O(\eta^{-\frac{1}{2}}) \right). \end{aligned} \quad (6.6)$$

If we fix a small $c_1 > 0$ and take $\eta > c_1^{-1}$, the product $2^k \eta^{\frac{m+p}{2}-k} \zeta^{m-p}$ will be maximal when $m+p$ takes its largest possible value k and also $m-p$ is as small as possible.

This means that $p = \lfloor \frac{k}{2} \rfloor$ and $m = k - \lfloor \frac{k}{2} \rfloor$. Moreover, if c_1 is small enough and η large enough, these values of m and p will make the expression (6.6) much larger than any other admissible values of m and p , in absolute value. We conclude that then

$$(-1)^{\lfloor \frac{k}{2} \rfloor} q_k \sim \eta^{-\frac{k}{2}} \zeta^{k-2\lfloor \frac{k}{2} \rfloor} \sim \begin{cases} \eta^{-\frac{k}{2}}, & k \text{ even} \\ c_1 \eta^{-\frac{k}{2}}, & k \text{ odd.} \end{cases} \quad (6.7)$$

We also need an estimate of the value of $p_t(x, y)$ for these x, y, t , and write the sum of the exponents of e in (6.1) as

$$\begin{aligned} -x_1 - y_1 - t - \frac{|x - y|^2}{4t} &= -\eta + O(1) - \frac{\eta}{2} + \frac{\zeta\sqrt{\eta}}{2} - \frac{\eta^2(1 + O(\eta^{-1}))}{2\eta(1 - \zeta/\sqrt{\eta})} \\ &= -\eta + O(1) - \frac{\eta}{2} + \frac{\zeta\sqrt{\eta}}{2} - \frac{\eta}{2} \left(1 + \frac{\zeta}{\sqrt{\eta}} + O(\eta^{-1}) \right) \\ &= -2\eta + O(1). \end{aligned}$$

Thus (6.1) implies that

$$p_t(x, y) \sim e^{-2\eta} t^{-\frac{n}{2}}.$$

We now combine this with (6.2) and (6.7), and take η large after fixing a small c_1 . Then (6.5) follows. \square

7 Proof of Theorem 3

For the local parts of the operators in Theorem 3, one can use the method of localization, since $\mathbb{R}^{(n,v)}$ has the local doubling property. Standard vector-valued singular integral theory then gives the weak type $(1, 1)$ of the local parts of g_k , h_k and H_k .

The parts at infinity of these operators can all be estimated by the method used for Theorem 1, in the following way.

Consider first the part at infinity of h_k , given by

$$h_k^\infty(f)(x) = \left(\int_0^{+\infty} \left| t^k \int_{|x-y|>1} \frac{\partial^k}{\partial t^k} p_t(x, y) f(y) d\mu(y) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

Using Minkowski's integral inequality, we get

$$h_k^\infty(f)(x) \leq \int_{|x-y|>1} |f(y)| \left(\int_0^{+\infty} \left| t^k \frac{\partial^k}{\partial t^k} p_t(x, y) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} d\mu(y).$$

We estimate the inner integral here.

Lemma 14 *We have*

$$\left(\int_0^{+\infty} \left| t^k \frac{\partial^k}{\partial t^k} p_t(x, y) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \lesssim |x - y|^{\frac{k-n}{2} - \frac{1}{4}} e^{-x_1 - y_1 - |x-y|}, \quad |x - y| > 1.$$

Since this implies a kernel estimate like (2.3) with q replaced by $k + 1/2$, we can argue as in Section 3 using the operator $\tilde{\mathcal{V}}_{k+1/2}$. In this way, the weak type $(1, 1)$ of h_1 and also (1.5) follow from Lemma 14.

Proof of Lemma 14. Combining (6.4) and (1.9), we conclude

$$\begin{aligned} \int_0^{+\infty} \left| t^k \frac{\partial^k}{\partial t^k} p_t(x, y) \right|^2 \frac{dt}{t} &\lesssim e^{-2x_1 - 2y_1 - 2|x-y|} \\ &\times \sum |x - y|^{4p} \int_0^{+\infty} \left| \frac{|x - y|}{2t} + 1 \right|^{2(m-p)} \left| \frac{|x - y|}{2t} - 1 \right|^{2(m-p)} t^{2m-4p-n} e^{-2t(\frac{|x-y|}{2t}-1)^2} \frac{dt}{t}, \end{aligned} \quad (7.1)$$

where the sum is taken over m and p such that $0 \leq p \leq [\frac{k}{2}]$ and $0 \leq m \leq k - p$. In the last integral, we estimate

$$\left| \frac{|x - y|}{2t} - 1 \right|^{2(m-p)} e^{-2t(\frac{|x-y|}{2t}-1)^2} \lesssim t^{p-m} e^{-t(\frac{|x-y|}{2t}-1)^2}. \quad (7.2)$$

Changing variables by $t = s|x - y|/2$, we see that the integral in the right-hand side of (7.1) is controlled by

$$\int_0^{+\infty} \left(\frac{1}{s} + 1 \right)^{2(m-p)} (|x - y|s)^{m-3p-n} e^{-\frac{1}{2}|x-y|s(\frac{1}{s}-1)^2} \frac{ds}{s}.$$

If restricted to the interval $1/2 < s < 2$, this integral will be of order of magnitude $|x - y|^{m-3p-n-1/2}$. The other parts of the integral are smaller because of the exponential decay. From this, Lemma 14 follows. \square

Let us now consider the part at infinity of g_k , $k \geq 1$. It is enough to verify the following estimate:

$$\left(\int_0^{+\infty} \left| t^k \frac{\partial^k}{\partial t^k} P_t(x, y) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \lesssim |x - y|^{-\frac{n+1}{2}} e^{-x_1 - y_1 - |x-y|}, \quad |x - y| > 1, \quad (7.3)$$

since we can then argue as in Section 3. For this we write the subordination formula as

$$P_t(x, y) = \int_0^{+\infty} \phi\left(\frac{t}{2\sqrt{u}}\right) p_u(x, y) \frac{du}{u} \quad \text{with} \quad \phi(s) = \frac{s}{\sqrt{\pi}} e^{-s^2}$$

and use Minkowski's integral inequality, getting

$$\begin{aligned}
& \left(\int_0^{+\infty} \left| t^k \frac{\partial^k}{\partial t^k} P_t(x, y) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \\
& \leq \int_0^{+\infty} \left\{ \left(\int_0^{+\infty} \left| \left(\frac{t}{2\sqrt{u}} \right)^k \phi^{(k)} \left(\frac{t}{2\sqrt{u}} \right) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\} p_u(x, y) \frac{du}{u} \\
& \lesssim \int_0^{+\infty} p_u(x, y) \frac{du}{u}.
\end{aligned}$$

In view of (1.8), the last expression is majorized by constant times

$$e^{-x_1-y_1} \int_0^{+\infty} u^{-\frac{n}{2}} e^{-u-\frac{|x-y|^2}{4u}} \frac{du}{u}.$$

By Lemma 6 (b), we see that for $|x-y| > 1$ this quantity is no larger than constant times $e^{-x_1-y_1-|x-y|} |x-y|^{-(n+1)/2}$, and (7.3) follows.

The next lemma will imply the weak type $(1, 1)$ of H_1 and also (1.6).

Lemma 15 *For $|x-y| > 1$, $t > 0$ and $k \geq 1$,*

$$\left| t^k \frac{\partial^k}{\partial t^k} p_t(x, y) \right| \lesssim e^{-x_1-y_1-|x-y|} |x-y|^{\frac{k-n}{2}}.$$

Proof. From (6.4) and (1.9), we obtain

$$\begin{aligned}
& \left| t^k \frac{\partial^k}{\partial t^k} p_t(x, y) \right| \\
& \lesssim e^{-x_1-y_1-|x-y|} \sum \left| \frac{|x-y|}{2t} - 1 \right|^{m-p} \left(\frac{|x-y|}{2t} + 1 \right)^{m-p} |x-y|^{2p} t^{m-2p-\frac{n}{2}} e^{-t(\frac{|x-y|}{2t}-1)^2},
\end{aligned}$$

the sum taken over $0 \leq p \leq [\frac{k}{2}]$, $0 \leq m \leq k-p$. We now use (7.2), where we first take square roots, and write again $t = s|x-y|/2$. The result will be

$$\left| t^k \frac{\partial^k}{\partial t^k} p_t(x, y) \right| \lesssim e^{-x_1-y_1-|x-y|} \sum \left(\frac{1}{s} + 1 \right)^{m-p} |x-y|^{\frac{p+m-n}{2}} s^{\frac{m-3p-n}{2}} e^{-\frac{1}{4}|x-y|s(\frac{1}{s}-1)^2}.$$

Here we simply delete the factor $|x-y|$ in the last exponent. Since $p+m \leq k$, we will get at most the quantity in the right-hand side of the lemma, multiplied by a bounded function of s .

Lemma 15 is proved. \square

To verify the sharpness parts of Theorem 3, we use Lemma 13. For h_k we get with $x \in \Sigma_\eta$

$$\left(h_k \left(\chi_{B(0,1)} \right) (x) \right)^2 \gtrsim \int_{\eta(1-2c_1/\sqrt{\eta})/2}^{\eta(1-c_1/\sqrt{\eta})/2} e^{-4\eta} \eta^{k-n} \frac{dt}{t} \gtrsim e^{-4\eta} \eta^{k-n-\frac{1}{2}}.$$

As in the last part of Section 3, this implies that h_k is not of weak type $(1, 1)$ for $k \geq 2$ and that (1.5) is sharp.

The case of H_k is even simpler, and this ends the proof of Theorem 3. \square

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References

- [1] G. Alexopoulos, An application of homogenization theory to harmonic analysis: Harnack inequalities and Riesz transforms on Lie groups of polynomial growth, *Canad. J. Math.* 44 (1992) 691–727.
- [2] A. Andersson, P. Sjögren, Ornstein-Uhlenbeck theory in finite dimension, Lecture Notes, Dept. Math. Sciences, Chalmers and University of Gothenburg 2012.
- [3] J.-P. Anker, Sharp estimates for some functions of the Laplacian on noncompact symmetric spaces, *Duke Math. J.* 65 (1992) 257–297.
- [4] J.-P. Anker, E. Damek, C. Yacoub, Spherical analysis on harmonic AN groups, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 23 (1996) 643–679.
- [5] P. Auscher, T. Coulhon, X.-T. Duong, S. Hofmann, Riesz transform on manifolds and heat kernel regularity, *Ann. Sci. École Norm. Sup. (4)* 37 (2004) 911–957.
- [6] D. Bakry, Étude des transformations de Riesz dans les variétés riemanniennes à courbure de Ricci minorée, *Séminaire de Probabilités XXI*, Lecture Notes in Math., vol. 1247, Springer, Berlin, 1987, pp. 137–172.
- [7] G. Carron, T. Coulhon, A. Hassell, Riesz transform and L^p -cohomology for manifolds with Euclidean ends, *Duke Math. J.* 133 (2006) 59–93.
- [8] L. Chen, T. Coulhon, J. Feneuil, E. Russ, Riesz Transform for $1 \leq p \leq 2$ Without Gaussian Heat Kernel Bound. *J. Geom. Anal.* DOI 10.1007/s12220-016-9728-5
- [9] T. Coulhon, X. T. Duong, Riesz transforms for $1 \leq p \leq 2$, *Trans. Amer. Math. Soc.* 351 (1999) 1151–1169.
- [10] T. Coulhon, X.-T. Duong, X.-D. Li, Littlewood-Paley-Stein functions on complete Riemannian manifolds for $1 \leq p \leq 2$, *Studia Math.* 154 (2003) 37–57.

- [11] T. Coulhon, H.-Q. Li, Estimations inférieures du noyau de la chaleur sur les variétés coniques et transformée de Riesz, *Arch. Math.* 83 (2004) 229–242.
- [12] A.F.M. ter Elst, D. W. Robinson, A. Sikora, Riesz Transforms and Lie Groups of Polynomial Growth, *J. Funct. Anal.* 162 (1999) 14–51.
- [13] E. B. Fabes, C. E. Gutiérrez, R. Scotto, Weak-type estimates for the Riesz transforms associated with the Gaussian measure, *Rev. Mat. Iberoam.* 10 (2) (1994) 229–281.
- [14] G. I. Gaudry, T. Qian, P. Sjögren, Singular integrals associated to the Laplacian on the affine group $ax + b$, *Ark. Math.* 30 (1992) 259–281.
- [15] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products*. 7th edition. Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger. Academic Press, Inc., San Diego, CA, 2007. Reproduction in P.R.China authorized by Elsevier (Singapore) Pte Ltd.
- [16] A. Grigor'yan, *Heat kernel and analysis on manifolds*. AMS/IP Studies in Advanced Mathematics, 47. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009.
- [17] W. Hebisch, T. Steger, Multipliers and singular integrals on exponential growth groups, *Math. Z.* 245 (2003) 37–61.
- [18] H.-Q. Li, La transformation de Riesz sur les variétés coniques, *J. Funct. Anal.* 168 (1999) 145–238.
- [19] H.-Q. Li, P. Sjögren, Weak type $(1, 1)$ for some operators related to the Laplacian with drift on real hyperbolic spaces. To appear in *Potential Anal.* DOI 10.1007/s11118-016-9590-x
- [20] H.-Q. Li, P. Sjögren, Y.-R. Wu, Weak type $(1, 1)$ of some operators for the Laplacian with drift, *Math. Z.* 282 (2-3) (2016) 623–633.
- [21] N. Lohoué, Comparaison des champs de vecteurs et des puissances du laplacien sur une variété riemannienne à courbure non positive, *J. Funct. Anal.* 61 (1985) 164–201.
- [22] N. Lohoué, Estimation des fonctions de Littlewood-Paley-Stein sur les variétés riemanniennes à courbure non positive, *Ann. Sci. École Norm. Sup.* (4) 20 (1987) 505–544.
- [23] N. Lohoué, Transformées de Riesz et fonctions sommables, *Amer. J. Math.* 114 (1992) 875–922.
- [24] N. Lohoué, S. Mustapha, Sur les transformées de Riesz sur les espaces homogènes des groupes de Lie semi-simples, *Bull. Soc. Math. France* 128 (2000) 485–495.
- [25] N. Lohoué, S. Mustapha, Sur les transformées de Riesz dans le cas du Laplacien avec drift, *Trans. Amer. Math. Soc.* 356 (2004) 2139–2147.

- [26] S. Pérez, F. Soria, Operators associated with the Ornstein-Uhlenbeck semigroup, J. London Math. Soc. (2) 61 (3) (2000) 857–871.
- [27] P. Sjögren, M. Vallarino, Boundedness from H^1 to L^1 of Riesz transforms on a Lie group of exponential growth, Ann. Inst. Fourier 58 (2008), 1117–1151.
- [28] E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*. Ann. of Math. Stud., vol. 63, Princeton Univ. Press, Princeton, N.J., 1970.
- [29] E. M. Stein, N. J. Weiss, On the convergence of Poisson integrals, Trans. Amer. Math. Soc. 140 (1969), 35–54.